This indicates that in the first place the circles (4) and (7), and (5) and (8) must be orthogonal. Secondly, for every value of λ there must be

$$2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2=0.$$

This equation is satisfied if the circles (5) and (7), and (4) and (8) are orthogonal. In this case the pencils (6) and (9) are said to be conjugate pencils of circles. Every circle of the one pencil is orthogonal to every other circle of the other. But in equation (12) we do not desire to change the points A, B, C, D. Thus, the equation can be satisfied by changing the radii of the circles (4), (5), (7), (8), which gives for the solutions of M_1, M_2, N_1, N_2 three equations with four unknown quantities. We can however fix one of the circles without altering the final result. This implies the theorem :

Two pencils of circles can be made projective in one and only one way such that corresponding circles in the projectivity are orthogonal.

The product of these projective pencils is a bicircular curve of the fourth order, as it is well known. In figure 2, we consider the two pencils of circles through A and B, and C' and D', where C' and D' are assumed to be imaginary and on the line l and in these pencils two orthogonal circles U and V' intersecting each other in two points J and J'. In these points draw tangent circles to U having their centers on l. These circles are orthogonal to V' and intersect each other in two fixed points C and D, *i. e.*, they belong to the conjugate pencil of circles of the pencil through C' and D'. Whence the general theorem :

The locus of the points of tangency of each two tangent-circles of two pencils of circles is a bicircular curve of the fourth order. The same curve is also produced by one of the pencils and the projective conjugate pencil of the other pencil.

Under the given conditions the equation of the curve may be written $U_1V_2'-U_2V_1'=0$.

It is easily seen that this curve passes through the four points A, B, C, D and as stated in the theorem contains the circular points at infinity as double-points.

5. Without entering into further details on the nature of this curve it may be mentioned that there exists an interesting connection between this curve and the circular curves of the third order if these are considered as locii of points from which two sects AB and CD appear under the same angle. An analogon exists in space, the discussion of which however goes over the limits of this article. A paper on this subject by the author was read in the January session of the Kansas Academy of Science and will appear in the next volume of the transactions of this Academy.

PROBLEMS.

1. Given n straight lines in a plane. Another straight line in this plane revolves about a fixed point and in every position intersects the n lines

in n points. These points determine n "sects" measured from the fixed point and their algebraic sum represents a point on the revolving line. What is the curve which this point describes?

2. Find a geometrical construction for the following problem: Given the distances AO, BO, CO of the points A, B, C of an equilateral triangle from a fixed point O. Construct the equilateral triangle, or triangles satisfying these conditions.

3. What is the locus of the points from which any two sects in space AB and CD (not in the same plane) appear under the same constant angle?

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from April Number.]

PROPOSITION XXIV. *The same hypothesis remaining; I say the four angles together (Fig. 27.) of the quadrilateral KDHK nearer the base AB are less (in hypothesis of acute angle) than the four angles together of the quadrilateral KHLK more remote from the same base; and indeed this is so, whether those two AX, BX somewhere at a finite distance meet toward the parts of the point X; or never meet one another; but toward those parts either ever more mutually approach each other, or somewhere receive a common perpendicular, after which of course (in accordance with Cor. II. of the preceding proposition) toward the same parts they begin mutually to separate.*

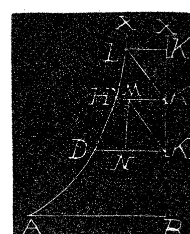

Fig. 27.

PROOF. Here however we suppose the portions KK assumed to be mutually equal. Since therefore (from the preceding) the side DK is greater than the side HK, and similarly HK greater than the side LK, the portion MK in HK is assumed equal to LK, and in DK the portion NK equal to HK; and MN, MK, LK are joined, truly the intermediate point K with the point L, and the point K near to the point B with the point M.

Now I proceed thus.

Since indeed the sides of the triangle KKL (I make beginning always from the point K nearer the point B) are equal to the sides of the triangle KKM, and the included angles equal, as being right, equal also will be (from Eu. I. 4) the bases LK, MK, and likewise equal the angles which correspond mutually,